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WAVE COALESCENCE AND ENTRAINMENT IN VERTICAL ANNULAR TWO-PHASE FLOW

N. S. WILKES,† B. J. AZZOPARDI† and C. P. THOMPSON‡
AERE Harwell, Oxon, England

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Abstract—A rigorous model for wave coalescence has been derived. The wave coalescence process has also been modelled by a Monte-Carlo technique. The results of the theories are in general in good agreement with the available experimental data. It had been noted that coalescence of two waves was accompanied by a large burst of entrainment. The above coalescence theory has been used to calculate that component of entrainment that is due to coalescence. Comparison of this and experimental data shows that the entrainment due to coalescence can be a significant portion of the total entrainment.

1. INTRODUCTION

Annular two phase flow is an important regime of two phase flow. It is characterised by a film of liquid on the surface of the pipe together with a fast flowing gas stream in the bulk of the pipe. Additionally there are droplets of liquid entrained in the gas stream. Here we shall be concerned with annular flow in the vertically upwards direction.

An important phenomenon in annular flow is the presence of waves on the surface of the liquid. These waves can be split up into two main groups: firstly, there are small, slow moving ripples which do not have a continuous life, and, secondly, there are disturbance waves which are faster and larger than the ripples, usually forming complete rings in the pipe and having a characteristically milky appearance. In this paper we shall only be concerned with the disturbance waves.

By measuring the velocities of waves passing a given point in the pipe, and also the time separation of successive waves passing that point, probability density functions for the velocity and time separation of waves at that point can be built up. Hall Taylor *et al.* (1963) have observed that disturbance waves tend to move with constant velocity and that if a faster wave overtakes a slower wave, then the two waves coalesce and usually continue with the speed of the faster wave.

Here it is our intention under these two simple hypotheses to determine theroretically the variation of the probability density functions for velocity and time separation of the disturbance waves with distance along the pipe. Although this can be done, so that analytical expressions can be determined for the probability density functions, they are in general not computable. For this reason we also develop a Monte-Carlo method, essentially a numerical experiment, which moves waves according to our hypotheses and furnishes us with the probability density functions for different stations in the pipe.

This problem has been discussed previously by Hall Taylor & Nedderman (1968) and by Azzopardi (1979). Under the same hypotheses, they used an approximate theory in which a wave could only be caught by the wave immediately behind it in order to find the variation of mean time separation and hence also frequency with distance along the pipe. Unfortunately due to internal inconsistencies in their theory, namely that the frequency calculated as the reciprocal of the mean time separation is different from the frequency calculated as the initial frequency multiplied by the proportion of waves not overtaken, they

†Engineering Sciences Division.
‡Computer Science and Systems Division.

obtained results differing by a considerable amount from the results to be presented here.

Hall Taylor *et al.* (1963) also observed that when two waves coalesced, the merging was accompanied by a sudden burst of entrainment. Using the additional simple hypotheses that in a coalescence all the volume of the slower wave is entrained, and that there is a simple relationship between the velocity and volume of a wave, we can also predict either analytically or through the Monte-Carlo method the expected amount of entrainment due to coalescence of disturbance waves at all distances along the pipe. The assumption that in a coalescence all of the slower wave is entrained can be justified as follows. Firstly, waves have been observed to travel at a velocity proportional to their height (see Azzopardi 1979 for collected evidence). Also the time–distance plots of Hall Taylor *et al.* (1963) show that when waves coalesce the remaining wave continues at its former speed. Any increase in its volume and therefore height would cause the velocity to be different to that before coalescence; therefore all the slower wave must have been entrained.

In section 2, we describe the analytical determination of the probability density functions and of the entrainment due to coalescence, and in section 3 we repeat the process with the Monte-Carlo method. In section 4, we describe the results and the comparison with experimental measurements. Finally, in section 5 we present the conclusions and discuss possible extensions to the work.

2. ANALYTIC THEORY

Our two major assumptions are that individual waves travel with constant velocity and that if a faster wave catches a slower wave, then the faster wave continues with the same velocity but the slower wave is completely entrained.

Given the initial probability density function $p(v)$ for wave velocity and $q(t)$ for time separation of waves, we shall proceed to determine theoretically these distributions for different stations along the pipe.

We shall at all times consider a cut-off normal distribution for the initial wave velocity distribution, i.e.

$$p(v) = \frac{\sqrt{2}}{\sqrt{\pi}\sigma\left(1 + \operatorname{erf}\dfrac{\bar{C}}{\sqrt{2}\sigma}\right)} \exp\left(-\frac{(v-\bar{C})^2}{2\sigma^2}\right). \qquad [2.1]$$

Here $\bar{C}$ and σ are respectively the mean and standard deviation of the non cut-off normal distribution, and if $\bar{C}/\sigma$ is large they will also very nearly be the mean and standard deviation of the actual distribution. We shall, however, discuss different distributions for the initial time separation of waves.

The calculation of the probability distributions for velocity and time separation at distance z along the pipe is quite lengthy and so the details have been deferred to the appendices. Appendix A describes the theory for an arbitrary initial time separation distribution $q(t)$, while appendices B and C describe the simplifications possible when $q(t)$ is either a δ-function or an exponential distribution.

For a wave of speed v, we calculate the probability of being overtaken, $PO(v, z)$, by distance z along the pipe and $P(v, t, z)$, the probability that the time separation at z with the following wave is greater than t.

The probability distribution $P(v, z)$ for wave velocities at station z is then given by

$$P(v, z) = \frac{p(v)[1 - PO(v, z)]}{\displaystyle\int_0^{\infty} p(v)[1 - PO(v, z)]\,\mathrm{d}v} \qquad [2.2]$$

and the frequency $f(z)$ of waves at distance z along the pipe is given by

$$f(z) = f_0 \int_0^\infty p(v)[1 - PO(v, z)]\, dv \qquad [2.3]$$

where f_0 is the initial frequency of waves.

Further the distribution of time separations is given by the following equation for $P(t, z)$ the probability that the time separation with the next wave is greater than t:

$$P(t, z) = \frac{\int_0^\infty p(v)P(t, v, z)\, dv}{\int_0^\infty p(v)[1 - PO(v, z)]\, dv}. \qquad [2.4]$$

To calculate the entrainment due to coalescence, we assume that when a wave is caught up it is instantaneously entrained. We also assume that the volume of a wave is directly proportional to the height of the wave squared, i.e.

$$\text{Volume} = Ch(v)^2 \qquad [2.5]$$

and the height of a wave is determined from its velocity v through a linear relation

$$h(v) = Av + B \qquad [2.6]$$

where the constants A, B and C depend only upon the bulk flow properties and are found from the data of Azzopardi (1979) and Azzopardi *et al.* (1979).

The expected total entrainment $ET(z)$ of liquid per unit circumference of pipe by distance z is then given by

$$ET(z) = f_0 \int_0^\infty C\rho p(v)[h(v)]^2 PO(v, z)\, dv \qquad [2.7]$$

where ρ is the density of the liquid and the entrainment rate $E(z)$ is given by

$$E(z) = \frac{d}{dz} ET(z). \qquad [2.8]$$

3. NUMERICAL SIMULATION

In the previous section, we obtained analytic expressions for the variation with distance along the pipe of the probability distributions of wave velocity and time separation for general initial distributions of time separation. However, except for the special cases of δ-function or exponential distributions for the initial time separation, these analytic expressions include integrals of arbitrary orders and are thus not computable. For this reason we have developed a Monte-Carlo method to generate distributions of wave velocity and time separation for general initial distributions of time separation including especially the physically interesting case of initial cut-off normal distribution of time separation.

The Monte-Carlo method is , broadly speaking, the numerical analogue of an actual experiment. The initial distributions of velocity and time separation are calculated from a random number generator and the arrival of the waves is monitored at a number of downstream points. At each observation point the order of arrival of waves is monitored and any wave which arrives after a wave that started behind it must have coalesced and is disregarded. New distributions of velocity and time separation are formed with the waves that survive. The whole process is thus reduced to one of sorting.

Preliminary calculations were done with a δ-function distribution of time separations and a cut-off normal distribution of velocity. The normally distributed random numbers were generated by the Harwell Subroutine Library program FAØ3A. These results were in good agreement with the analysis of the previous section.

Further calculations were carried out for the physically interesting case of cut-off normal distributions of both the time separation and the velocity. These results are given in the next section.

The method is extremly flexible (and also quite cheap). Almost any probability distribution can be specified quite easily and used for either the time separation or velocity distributions. Moreover it should be possible to incorporate slight modifications to the model such as wave repulsion, non-constant wave velocities, or wave coalescence if the separation is less than some tolerance. The most obvious disadvantages are two-fold. Firstly, the coalescence is a discrete phenomenon so an extremely large number of waves is required to give an accurate estimate of the entrainment which occurs in very short distances. (One also needs large numbers to ensure that a reasonable quantity of waves reach distant observation points.) The second difficulty is deciding when a sufficiently large number of waves have been monitored to give reliable results. It is sometimes possible, with Monte-Carlo methods, to give confidence intervals on the results but the problem proved too difficult for the present work. Instead the runs were repeated with increasing numbers of waves until a doubling of the number of waves gave rise to less than 1 percent change in frequency.

The Monte-Carlo method has been quite effective in simulating this preliminary work. It seems likely that similar techniques can be extended to study more complicated models, although one anticipates increases in complexity leading to rapidly increasing running costs.

4. RESULTS AND COMPARISON WITH EXPERIMENT

The analytical theories described in section 2 and the Monte-Carlo method described in section 3 can all describe the evolution of the distributed wave velocities and time separations from given initial distributions. However the initial distributions are not presently available. Therefore we are constrained to assuming initial distributions and comparing the results predicted along the tube with available experimental data. As was stated earlier, the velocity distribution was assumed to be normal. This assumption is consistent with the experimental results of Hall Taylor & Nedderman (1968). These workers also found that the standard deviation of wave velocities appeared to be almost independent of both gas and liquid flow rates. Therefore for any particular example only the initial mean wave velocity need be specified.

Less information is available concerning the initial distribution of time separations of waves. The mean initial time separation is the reciprocal of the mean initial frequency and reasonable estimates can be made for this. We only use δ-function, exponential and cut-off normal distributions here and the first two of these are completely specified by the mean. For the cut-off normal distribution the standard deviation is also required. As the standard deviation increases from zero to the mean value, the cut-off normal distributions varies between the δ-function and exponential distributions. To obtain a distribution significantly different from these two limits we have therefore used a standard deviation of 0.4 times the mean for the cut-off normal distribution.

An example of the variation of mean frequency along the tube is shown in figure 1. The conditions selected are gas mass flux = 79.4 kg/m^{2}s and liquid mass flux = 79.4 kg/m^{2}s.

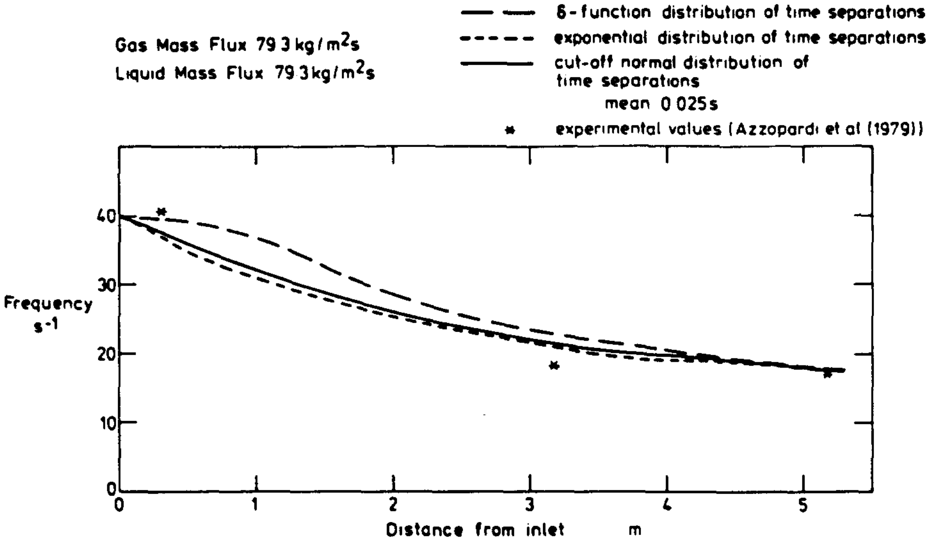

Figure 1. Variation of wave frequency with distance.

Figure 1 also shows experimental values taken from Azzopardi *et al.* (1979). As can be seen the agreement is quite reasonable. The ratio of mean frequency at any position along the tube to the initial frequency can be shown to be a strong function of the group $f_0 z/\bar{C}^2$, where $\bar{C}$ is the mean initial wave velocity, and only a weak function of f_0 and $\bar{C}$ separately. Figure 2 shows some of the data of Azzopardi *et al.* (1979) plotted as $f_0/f(z) - 1$ vs $f_0 z/\bar{C}^2$, also shown are the predictions of the above theories. The agreement is again quite reasonable—some of the scatter can be attributed to experimental uncertainty.

Brown (1978) has measured the variation of mean wave velocities along the tube. His results for gas mass flux of 79.4 kg/m^{2}s and liquid mass flux of 79.4 kg/m^{2}s are compared with the predictions of theories in figure 3. Initial mean and standard deviation of wave velocities of 3.0 and 0.15 m/s and an initial mean frequency of 40 Hz were assumed. The predictions can be seen to be very close to the experimental values, the difference being comparable to the experimental scatter.

Figure 4 shows a comparison between theoretical and measured wave velocity distributions. The measured values were taken from Hall Taylor & Nedderman (1968). The theoretical distribution curve is taken from a case with an initial δ-function distribution of time separations of mean 0.025 s. The normal initial wave distribution had a mean of 3.4 m/s and a standard deviation of 0.15 m/s. The theoretical distribution was calculated for z of 3 m whereas the experimental results were determined from an inspection of distance/time plots for waves between 1 and 6 m from the inlet. The agreement is again most reasonable with the skewness evident in the experimental values being well reproduced by the theory.

Time separation distribution data has been abstracted from film thickness traces taken by Azzopardi *et al.* (1979). These values are given in figure 5 together with the theoretical curve for an initial δ-function distribution of time separations. The mean of the δ-function was 0.1 s and the normal initial velocity distribution had mean and standard deviation of 2.25 and 0.15 m/s respectively. Here the theory is in less good agreement with the experimental data than for the previous figures. Experimentally very few waves are observed close together which could be explained by the fact that if waves are close together they overlap or by the fact that waves accelerate when close behind another wave. It would, however, be inconsistent for the theory to predict few small time separations as the amount of coalescence at any time is directly proportional to the number of waves with zero time separation.

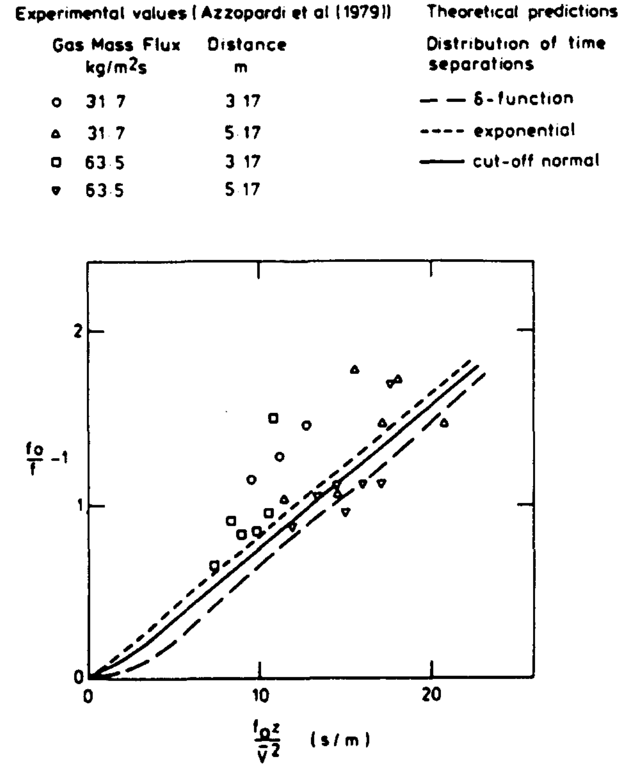

Figure 2. Variation of non-dimensionalized mean time separation with distance.

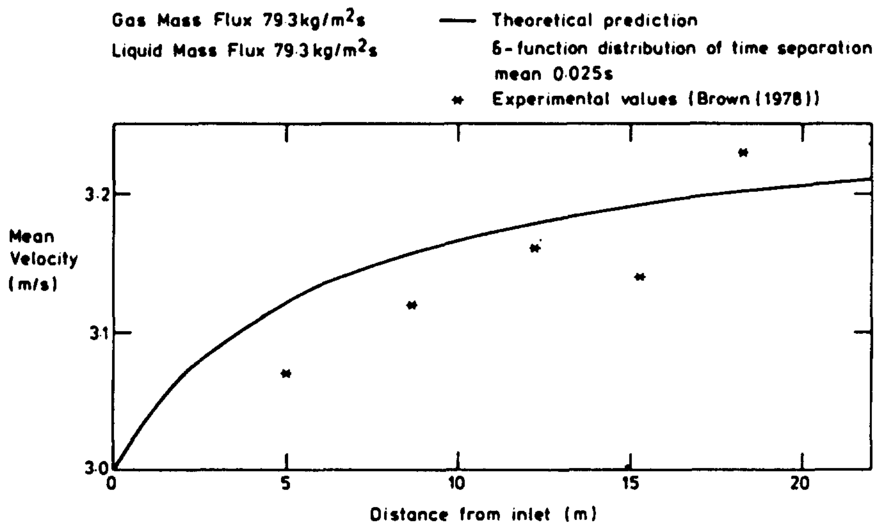

Figure 3. Variation of mean wave velocity with distance.

The evolution of the velocity and time separation distributions predicted by the theory are shown in figures 6 and 7 respectively. Both figures refer to the same flow rates and are for an inital δ-function distribution of time separations of mean 0.025 s and an initial normal velocity distribution of mean and standard deviation of 3.4 and 0.15 m/s respectively. Other shapes of initial time separation give very similar development for both

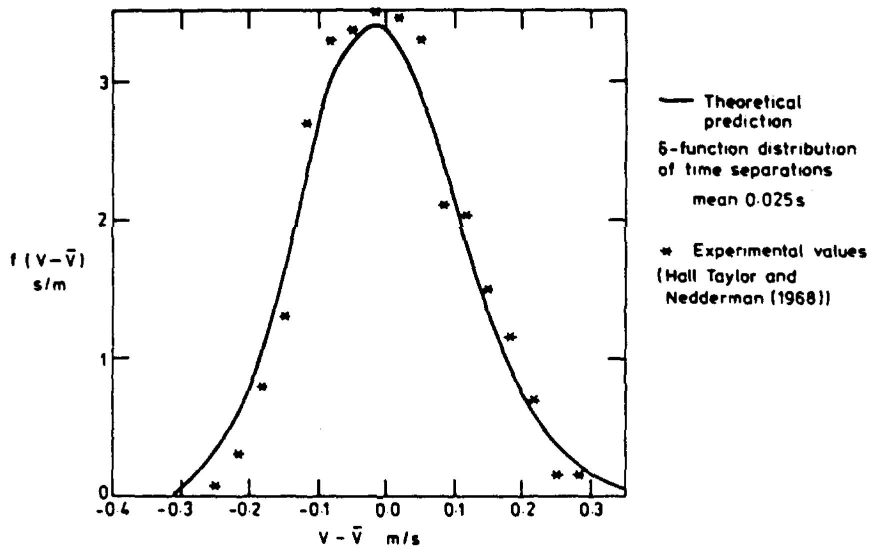

Figure 4. Probability distribution of wave velocities.

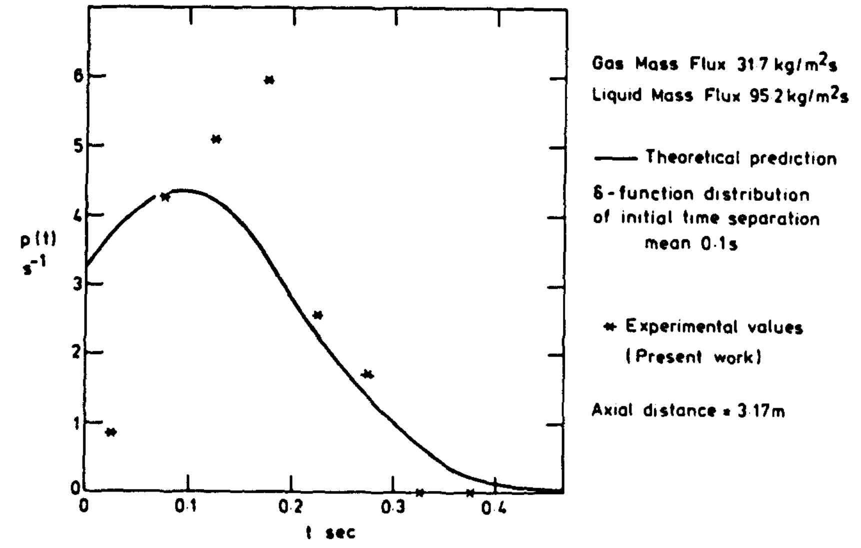

Figure 5. Probability distribution of time separations.

velocity and time separation distributions. This latter fact is surprising, the exponential distribution stays exponential whereas the δ-function and normal distribution evolve to exponential form.

Thus apart from the large fraction of small time separations predicted by the theories and not observed in experiments the theories are very good at showing the behaviour of wave frequencies and velocities along the tube. The theories therefore will be used to examine the split in the entrainment rate between that due to coalescence and that present even at equilibrium. The results of the calculations of that portion of entrainment due to coalescence are shown in figure 8. Curves are given for the analytical theory starting from both a δ-function and an exponential distribution and for the Monte Carlo analysis starting from a cut off normal distribution. This part of the analysis also uses the following

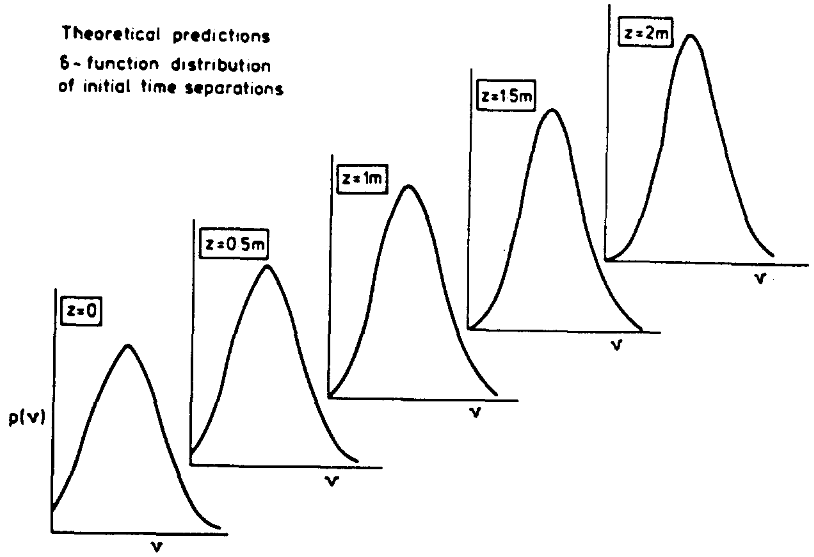

Figure 6. Variation with distance of wave velocity distribution.

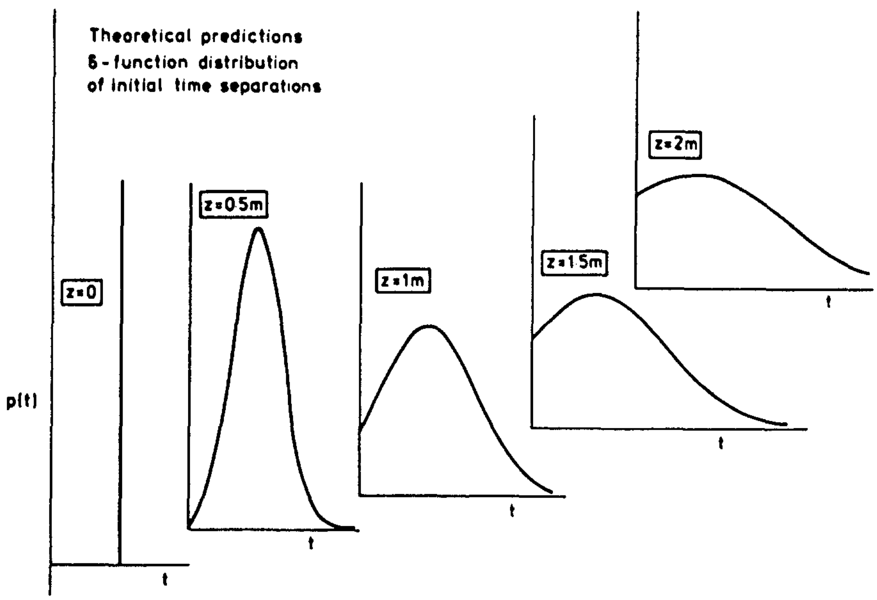

Figure 7. Variation with distance of time separation distribution.

relationships:

$$\bar{V} = 22\pi h(v)^2 \quad [4.1]$$

$$h(v) = \sqrt{20}\, 10^{-4}(v - 1). \quad [4.2]$$

The former is taken from the volume per wave data of Azzopardi *et al.* (1979) and the wave height data of Hewitt & Nicholls (1969). The latter is fitted to the data of Hewitt and Nicholls as replotted by Azzopardi (1979). The initial wave frequency used was 40 Hz, the initial velocity distribution was characterised by a mean of 3.4 m/s and a standard

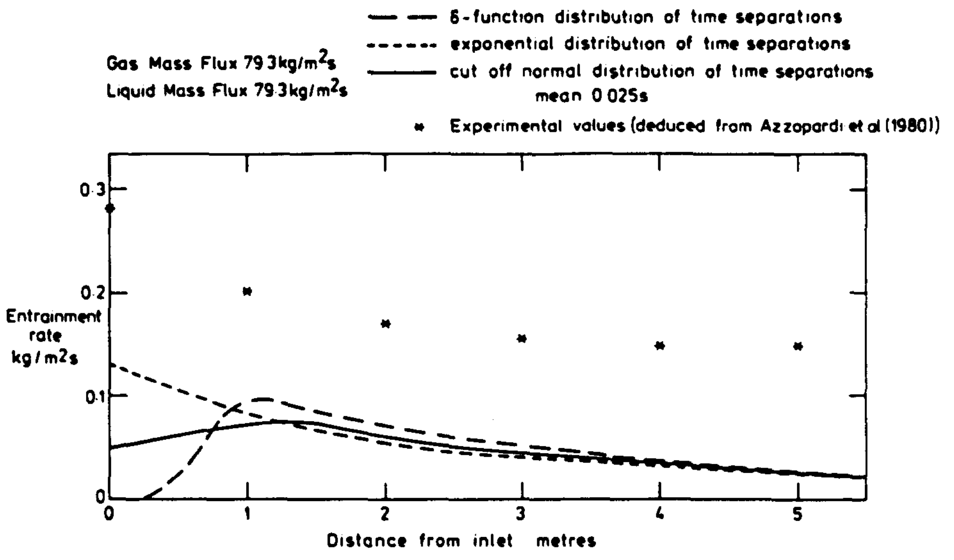

Figure 8. Variation of entrainment rate with distance.

deviation of 0.15 m/s. Values of total entrainment rates have been deduced from a mass balance on the entrained liquid

$$\frac{dG_{LE}}{dz} = \frac{4}{d}(E - D) \qquad [4.3]$$

where G_{LE} is the entrained liquid mass flux, E is the entrainment rate and D is deposition rate. It is further assumed that

$$D = \frac{kG_{LE}}{G_G} \qquad [4.4]$$

where k is the mass transfer coefficient and G_G is the gas mass flux. Combining [4.3] and [4.4] gives an expression for the entrainment rate in terms of the entrained liquid mass flux and its gradient. The data of Azzopardi *et al.* (1980) for entrained liquid mass flux at different positions along the tube was used to determine the entrainment rate.

The $G_{LE}(z)$ data was fitted with a simple function and then the function was differentiated. The resulting entrainment data is shown in figure 8. The results here show that the coalescence due to entrainment is a significant portion of the total entrainment is some parts of the tube irrespective of which initial distribution is considered. Hall Taylor (1966) suggested that the entrainment rate could be described by

$$E = af - b\frac{\partial f}{\partial z} \qquad [4.5]$$

where the second term describes the entrainment due to coalescence and the first term refers to background entrainment which we can expect to be proportional to the number of waves present. The three curves shown in figure 8 have been used to derive values for the second term in [4.5] above, experimental results were used for E. Values of the constant a were then sought. In all three cases a was found to vary with distance along the tube.

This item is the subject of further work.

5. CONCLUSIONS

From the work presented above it is possible to draw the following conclusions:

(1) A rigorous theory for the coalescence of waves in annular two phase flow has been derived. This allows for the coalescence of neighbouring waves and for the coalescence of the resulting wave with its neighbours. The results of the theory give reasonable agreement with available data.

(2) The theory can be extended to give that portion of entrainment due to coalescence. The calculations show that at some points in the tube the extrainment due to coalescence is a significant portion of the total entrainment. However, the remaining entrainment has not been found to be proportional to the local wave frequency.

One drawback of the current analysis is that the initial distribution of time separations and the initial distribution of wave velocities must be supplied. A possibility for future work is to determine these distributions from time distance plots taken from side view cine films. Further work on distinguishing the coalescence and background entrainment is also required. It might be possible to do this by creating single artificial waves using the technique of Azzopardi & Whalley (1980) at various frequencies and noting the effect of distance and frequency.

NOMENCLATURE

a	constant in [4.5], (kg/m^2)
b	constant in [4.5], (kg/m)
$\bar{C}$	mean wave velocity, m/s
D	deposition rate, kg/m^{2}s
$E(z)$	entrainment rate, kg/m^{2}s
$ET(z)$	total entrained fluid, kg/ms
$f(z)$	frequency, s^{-1}
f_0	initial frequency, s^{-1}
G_{LE}	entrained liquid mass flux, kg/m^{2}s
$h(v)$	wave height, m
k	deposition coefficient, m/s
$p(v)$	initial wave velocity density function, s/m
$PO(v, z)$	probability of wave of speed v having been overtaken by distance z
$P(v, t, z)$	probability that time separation between wave of speed v and the following wave at distance z along the pipe is greater than t
$P(t, z)$	probability that time separation between any two waves at distance z along the pipe is greater than t
$q(t)$	initial time separation distribution, s^{-1}
t	time separation, s
v	wave velocity, m/s
z	distance along pipe, m
ρ	fluid density, kg/m^3
σ	standard deviation of initial velocity distribution, m/s

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APPENDIX A

For a wave of speed v, the probability $PO(v, z)$ of having been overtaken at a distance z along the pipe is given by

$$PO(v, z) = \sum_{n=1}^{\infty} PO_n(v, z).$$

Here $PO_1(v, z)$ is the probability of having been overtaken by the wave behind: $PO_2(v, z)$ is the probability of having been overtaken by the wave 2 behind but not by the wave immediately behind: PO_3 is the probability of having been overtaken by the wave 3 behind but not by the two waves immediately behind, and so on.

The wave of speed v is overtaken by the wave behind if its speed V_1 is such that

$$\frac{z}{v} > \tau_1 + \frac{z}{V_1}$$

where τ_1 is the initial time separation of the two waves. The probability of having been overtaken by the wave behind is thus given by

$$PO_1(v, z) = \int_0^{z/v} q(\tau_1) g\left[\frac{zv}{z - v\tau_1}\right] d\tau_1$$

where $g(x)$ is defined by

$$g(x) = \int_x^{\infty} p(v)\, dv.$$

The wave of speed v is overtaken by the wave 2 behind but not by the wave immediately behind if their velocities V_2 and V_1 respectively satisfy

$$\frac{z}{v} > \tau_1 + \tau_2 + \frac{z}{V_2}$$

and

$$\frac{z}{v} < \tau_1 + \frac{z}{V_1}$$

where τ_1, τ_2 are the initial time separations between the first and second, and second and third waves respectively. The probability $PO_2(v, z)$ of having been overtaken by the wave 2 behind but not the wave immediately behind is thus given by

$$PO_2(v, z) = \int_0^{z/v} \int_0^{(z/v) - \tau_1} q(\tau_1)q(\tau_2)g\left[\frac{zv}{z - v\tau_1 - v\tau_2}\right]\left[1 - g\left[\frac{zv}{z - v\tau_1}\right]\right] d\tau_2\, d\tau_1.$$

$PO_3(v, z)$ can also be obtained in a similar way and is given by

$$PO_3(v, z) = \int_0^{z/v} \int_0^{(z/v) - \tau_1} \int_0^{(z/v) - \tau_1 - \tau_2} q(\tau_1)q(\tau_2)q(\tau_3)g\left[\frac{zv}{z - v\tau_1 - v\tau_2 - v\tau_3}\right]$$

$$\times\left[1 - g\left[\frac{zv}{z - v\tau_1}\right]\right]\left[1 - g\left[\frac{zv}{z - v\tau_1 - v\tau_2}\right]\right] d\tau_3\, d\tau_2\, d\tau_1.$$

$PO_n(v, z)$ for higher values of n are also calculable and are nth order integrals.

The probability distribution of velocities a distance z along the pipe is then given by

$$P(v, z) = \frac{p(v)[1 - PO(v, z)]}{\int_0^{\infty} p(v)[1 - PO(v, z)]\, dv}.$$

Further the mean frequency $f(z)$ can immediately be derived to be

$$f(z) = f_0 \int_0^{\infty} p(v)[1 - PO(v, z)]\, dv$$

where f_0 is the initial mean frequency given by

$$f_0 = \left(\int_0^{\infty} \tau q(\tau)\, d\tau\right)^{-1}.$$

We can also proceed to calculate the probability density function of time separation at distance z along the pipe. This is found in a similar way to the probability of having been overtaken. Let $Q(t, v, z)$ be the probability that the time separation of a wave of speed v and the following wave is less than t at a distance z along the pipe. Then $Q(t, v, z)$ is given by

$$Q(t, v, z) = \sum_{n=1}^{\infty} Q_n(t, v, z)$$

where $Q_1(t, v, z)$ is the probability that the time separation with the wave immediately behind initially is less than t, $Q_2(t, v, z)$ is the probability that the time separation with the wave initially behind is greater than t, but the time separation with the wave initially 2 behind is less than t, but the time separation with the wave initially 2 behind is less than t, etc.

The time separation with the wave immediately behind initially is less than t if its speed V_1 is such that

$$\frac{z}{v} + t > \tau_1 + \frac{z}{V_1}$$

where τ_1 is again the initial time separation of the two waves. The probability that the time separation is less than t is thus

$$Q_1(t, v, z) = \int_0^{(z/v)+t} q(\tau_1) g\left[\frac{zv}{z + vt - v\tau_1}\right] d\tau_1.$$

Similarly we can calculate $Q_2(t, v, z)$ to be

$$Q_2(t, v, z) = \int_0^{(z/v)+t} \int_0^{(z/v)+t-\tau_1} q(\tau_1) q(\tau_2) g\left[\frac{zv}{z + vt - v\tau_1 - v\tau_2}\right] \left[1 - g\left[\frac{zv}{z + vt - v\tau_1}\right]\right] d\tau_2\, d\tau_1.$$

Also $Q_n(t, v, z)$ for n greater than 2 can be calculated to be nth order integrals.

On renormalisation to account for waves that have been overtaken and thus disappeared, it follows that the overall probability that the time separation at a distance z along the pipe is greater than t is givem by

$$P(t, z) = \frac{1 - \int_0^{\infty} p(v) Q(t, v, z)\, dv}{\int_0^{\infty} p(v)[1 - PO(v, z)]\, dv}.$$

APPENDIX B

In the special case when the probability density function of time separations is a delta function, that is waves are emitted at constant time intervals τ_0, the analysis can be simplified as follows.

As the initial positions of the waves are now independent, the probability $PNO(v, z)$ of a wave of speed v not having been overtaken at distance z along the pipe is the product of the probabilities of not having been overtaken by the individual waves, i.e.

$$PNO(v, z) = \prod_{n=1}^{\infty} PNO_n(v, z).$$

The wave is not overtaken by the wave n behind if its speed V_n is such that

$$\frac{z}{v} < n\tau_0 + \frac{z}{V_n}.$$

Hence $PNO_n(v, z)$ is given by

$$PNO_n(v, z) = P\left[V_n < \frac{zv}{z - n\tau_0 v}\right] \quad \text{if } v < \frac{z}{n\tau_0}$$

$$= 1 \quad \text{if } v \geqslant \frac{z}{n\tau_0}.$$

The probability density function of velocities at distance z down the pipe is then given by

$$p(v; z) = \frac{p(v)PNO(v, z)}{\int_{-\infty}^{\infty} p(v)PNO(v, z)\,\mathrm{d}v}$$

and the mean frequency $f(z)$ of waves at distance z along the pipe is

$$f(z) = \frac{\int_{-\infty}^{\infty} p(v)PNO(v, z)\,\mathrm{d}v}{\tau_0}.$$

To obtain the probability distribution of time separations at distance z along the pipe we note that for a wave of velocity v, the probability $P(t, v, z)$ that the time separation with all waves behind is greater than t is the product of the probabilities of time separation with individual waves behind is greater than t, i.e.

$$P(t, v, z) = \prod_{n=1}^{\infty} P_n(t, v, z).$$

The time separation with the wave n behind is greater than t if its speed V_n is such that

$$\frac{z}{v} + t < n\tau_0 + \frac{z}{V_n}.$$

Thus

$$P_n(t, v, z) = P\left[V_n < \frac{zv}{z + v(t - n\tau_0)}\right] \quad \text{if } t > n\tau_0 - \frac{z}{v}$$

$$= 1 \quad \text{if } t \leqslant n\tau_0 - \frac{z}{v}.$$

The overall probability that the time separation at a distance z along the pipe is greater than t is then given by

$$P(t, z) = \frac{\int_{-\infty}^{\infty} p(v)P(t, v, z)\,\mathrm{d}v}{\int_{-\infty}^{\infty} p(v)PNO(v, z)\,\mathrm{d}v}.$$

APPENDIX C

For the special case in which the initial probability density function of time separation is an exponential function the analysis of Appendix A is appreciably simplified.

We have the probability density function $f(\tau)$ given by

$$f(\tau) = \frac{1}{\bar{\tau}_0}\exp\left(-\frac{\tau}{\bar{\tau}_0}\right)$$

where $\bar{\tau}_0$ is the initial mean time separation.

In this case we have

$$PO_1(v,z)=\frac{1}{\bar{\tau}_0}\int_0^{z/v}\exp\left(-\frac{\tau_1}{\bar{\tau}_0}\right)g\left[\frac{zv}{z-v\tau_1}\right]\mathrm{d}\tau_1$$

Also

$$PO_2(v,z)=\frac{1}{\bar{\tau}_0^2}\int_0^{z/v}\int_0^{(z/v)-\tau_1}\exp\left(-\frac{\tau_1+\tau_2}{\bar{\tau}_0}\right)g\left[\frac{zv}{z-v\tau_1-v\tau_2}\right]\left[1-g\left[\frac{zv}{z-v\tau_1}\right]\right]\mathrm{d}\tau_2\,\mathrm{d}\tau_1$$

which on changing of the dummy variables in the integral can be written as

$$PO_2(v,z)=\frac{1}{\bar{\tau}_0^2}\int_0^{z/v}\exp\left(-\frac{s}{\bar{\tau}_0}\right)g\left[\frac{zv}{z-vs}\right]\int_0^s\left[1-g\left[\frac{zv}{z-v\tau}\right]\right]\mathrm{d}\tau\,\mathrm{d}s.$$

Similarly $PO_3(v,z)$ can be written as

$$PO_3(v,z)=\frac{1}{\bar{\tau}_0^3}\int_0^{z/v}\exp\left(-\frac{s}{\bar{\tau}_0}\right)g\left[\frac{zv}{z-vs}\right]\int_0^s\int_0^u\left[1-g\left[\frac{zv}{z-vu}\right]\right]\left[1-g\left[\frac{zv}{z-v\tau}\right]\right]\mathrm{d}\tau\,\mathrm{d}u\,\mathrm{d}s.$$

If we now observe that

$$\int_0^s\int_0^u\left[1-g\left[\frac{zv}{z-vu}\right]\right]\left[1-g\left[\frac{zv}{z-v\tau}\right]\right]\mathrm{d}\tau\,\mathrm{d}u=\frac{1}{2}\left(\int_0^s\left[1-g\left[\frac{zv}{z-v\tau}\right]\right]\mathrm{d}\tau\right)^2$$

and similarly for the higher order integrals in $PO_n(v,z)$, it follows that $PO(v,z)$ defined by

$$PO(v,z)=\sum_{n=1}^{\infty}PO_n(v,z)$$

$$=\frac{1}{\bar{\tau}_0}\int_0^{z/v}\exp\left(-\frac{s}{\bar{\tau}_0}\right)g\left[\frac{zv}{z-vs}\right]\exp\left(\frac{1}{\bar{\tau}_0}\int_0^s\left[1-g\left[\frac{zv}{z-v\tau}\right]\right]\mathrm{d}\tau\right)\mathrm{d}s$$

and this can be simplified to

$$PO(v,z)=1-\exp\left(-\frac{1}{\bar{\tau}_0}\int_0^{z/v}g\left[\frac{zv}{z-v\tau}\right]\mathrm{d}\tau\right).$$

The equations for the probability distribution of velocities and for mean frequency can be respectively written as

$$P(v,z)=\frac{p(v)\exp\left(-\frac{1}{\bar{\tau}_0}\int_0^{z/v}g\left[\frac{zv}{z-v\tau}\right]\mathrm{d}\tau\right)}{\int_0^{\infty}p(v)\exp\left(-\frac{1}{\bar{\tau}_0}\int_0^{z/v}g\left[\frac{zv}{z-v\tau}\right]\mathrm{d}\tau\right)\mathrm{d}v}$$

and

$$f(z) = \frac{1}{\bar{\tau}_0}\int_0^{\infty} p(v)\exp\left(-\frac{1}{\bar{\tau}_0}\int_0^{z/v} g\left[\frac{zv}{z - v\tau}\right]\mathrm{d}\tau\right)\mathrm{d}v.$$

Further the analysis leading to the probaility distribution of time separation also can be simplified. Using a similar analysis to the above, it is found that

$$Q(t, v, z) = 1 - \exp\left(-\frac{1}{\bar{\tau}_0}\int_0^{t+(z/v)} g\left[\frac{zv}{z + vt - v\tau}\right]\mathrm{d}\tau\right)$$

and hence

$$P(t, z) = \frac{\displaystyle\int_0^{\infty} p(v)\exp\left(-\frac{1}{\bar{\tau}_0}\int_0^{t+(z/v)} g\left[\frac{zv}{z + vt - v\tau}\right]\mathrm{d}\tau\right)\mathrm{d}v}{\displaystyle\int_0^{\infty} p(v)\exp\left(-\frac{1}{\bar{\tau}_0}\int_0^{z/v} g\left[\frac{zv}{z - v\tau}\right]\mathrm{d}\tau\right)\mathrm{d}v}.$$